Open Sentences and Free Variables in RPL

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We know what a sentence of RPL looks like. We can construct sentences via the following simple procedure: if φ is a sentence of RPL, and φ contains at least one singular term $(a, b, c, \ldots)$, then we can replace any term in φ with a variable that doesn't occur in φ and then stick a quantifier in front of φ with that variable.

EXAMPLE

We know that the following a sentence of RPL:

$$(\forall x)R^3abx$$

So by our rule, we can construct a new sentence by replacing any term in this sentence with a new variable, say 'y', and then bind 'y'. So the following count as sentences of RPL built from the previous one:

$$(\exists y)(\forall x)R^3ayx$$
$$(\exists y)(\forall x)R^3ybx$$
$$(\exists y)(\forall x)R^3yyx$$

The result is that all variables in a sentence must be bound. So the following doesn't count as a sentence:

$$(\exists y)(\forall x)R^3yzx$$

because 'z' isn't bound by any quantifier. But it is at least an *open sentence*.

DEFINITION (OPEN SENTENCE)

An **open sentence** is the result of replacing some of the singular terms in φ with variables that are not bound by any quantifier.

So things like 'R^2xy', 'F^1x', '$(\exists z)R^3xyz$', etc. count as *open* sentences. We could convert these into *closed* sentences by binding the necessary variables. For instance, '$(\forall x)(\forall y)(\exists z)R^3xyz$' is now a closed sentence of RPL.

Open sentences, strictly speaking, don't mean anything by themselves. Semantically, they're "incomplete". They're akin to fragments such as 'walks the dog' or 'Jim likes to'. We can't understand the meaning of an open sentence unless we replace the free variables with singular terms, or if we close the sentences by binding the free variables with quantifiers.

Definition (Free Variables)

A **free variable** v in an open sentence φ is a variable that's not bound by a quantifier. Every other variable of φ is a **bound variable**. For instance, 'y' is a free variable in '$(\forall x)(\exists z)R^3xyz$', while '$x$' and '$z$' are bound variables.

Notation: If v is the only free variable in φ, we may write " φ^v " to indicate this. If u and v are the only free variables in φ, we may instead write " $\varphi^{u,v}$ ". And so on. (Often, you'll see people write "$\varphi(v)$" instead of " φ^v ", "$\varphi(u,v)$" instead of " $\varphi^{u,v}$ ", etc.)

If φ^x is an open sentence with 'x' as its sole free variable, and if 'y' occurs nowhere in φ^x, then " φ^y " denotes the open sentence that results from replacing *every* instance of 'x' with 'y' in φ.

Example

Let's get some practice with the notation:

$$\begin{aligned}
\varphi^x = \text{`}F^1x\text{'} \quad &\Rightarrow \quad \varphi^y = \text{`}F^1y\text{'}\\
\varphi^x = \text{`}R^2ax\text{'} \quad &\Rightarrow \quad \varphi^y = \text{`}R^2ay\text{'}\\
\varphi^y = \text{`}(\forall x)R^2yx\text{'} \quad &\Rightarrow \quad \varphi^z = \text{`}(\forall x)R^2zx\text{'}\\
\varphi^y = \text{`}(\forall x)R^2yx\text{'} \quad &\Rightarrow \quad \varphi^x = \ ?\\
\varphi^{x,y} = \text{`}(\exists z)(F^1y \,\&\, G^2xz)\text{'} \quad &\Rightarrow \quad \varphi^{y,x} = \text{`}(\exists z)(F^1x \,\&\, G^2yz)\text{'}\\
\varphi^{x,y} = \text{`}(\exists z)(F^1y \,\&\, G^2xz)\text{'} \quad &\Rightarrow \quad \varphi^{x,x} = \text{`}(\exists z)(F^1x \,\&\, G^2xz)\text{'}
\end{aligned}$$

In the fourth example, it's tempting to say $\varphi^x = \text{`}(\forall x)R^2xx\text{'}$. But we're not allowed to use variables that already occur in φ^y, so the result is undefined.

We may also bind open sentences, such as:

$$\begin{aligned}
\varphi^x = \text{`}F^1x\text{'} \quad &\Rightarrow \quad \ulcorner(\forall x)\varphi^x\urcorner = \text{`}(\forall x)F^1x\text{'}\\
\varphi^x = \text{`}R^2ax\text{'} \quad &\Rightarrow \quad \ulcorner(\exists x)\varphi^x\urcorner = \text{`}(\exists x)R^2ax\text{'}\\
\varphi^y = \text{`}(\forall x)R^2yx\text{'} \quad &\Rightarrow \quad \ulcorner(\exists z)\varphi^z\urcorner = \text{`}(\exists z)(\forall x)R^2zx\text{'}\\
\varphi^{x,y} = \text{`}(\exists z)(F^1y \,\&\, G^2xz)\text{'} \quad &\Rightarrow \quad \ulcorner(\forall x)\varphi^{x,x}\urcorner = \text{`}(\forall x)(\exists z)(F^1x \,\&\, G^2xz)\text{'}
\end{aligned}$$