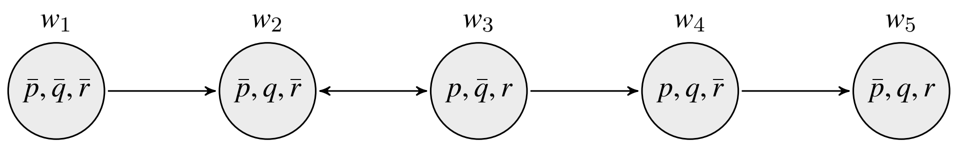
Counterfactuals

Phil 143 Worksheet

1. Using the truth definition for counterfactuals on slide 13, determine which of the following formulas below is true at $\mathcal{M}, w_1$.



The relevant ordering is given as follows:

w_1: $w_1 < w_2 \approx w_3 < w_4 < w_5$
w_3: $w_3 < w_2 < w_1 < w_4 < w_5$ (not drawn, only needed for (d))

(a) $\boxed{\text{True.}}$ $p \mathrel{\Box\!\!\rightarrow} r$
(b) $\boxed{\text{False.}}$ $(p \wedge q) \mathrel{\Box\!\!\rightarrow} r$
(c) $\boxed{\text{False.}}$ $\neg p \mathrel{\Box\!\!\rightarrow} (p \mathrel{\Box\!\!\rightarrow} q)$
(d) $\boxed{\text{False.}}$ $p \mathrel{\Box\!\!\rightarrow} (\neg p \mathrel{\Box\!\!\rightarrow} r)$

2. Show that $(p \mathrel{\Box\!\!\rightarrow} q) \vee (p \mathrel{\Box\!\!\rightarrow} \neg q)$ is valid on a world-ordering frame $\mathcal{F} = \langle W \{\leqslant_w\}_{w \in W} \rangle$ iff Stalnaker's assumptions holds on $\mathcal{F}$: for all nonempty $S \subseteq W$ and all $w \in W$, $|\mathbf{Min}_{\leqslant_w}(S)| = 1$. You may assume that every $\leqslant_w$ in $\mathcal{F}$ is well-founded.

► Solution: Recall the relevant definitions:

$$\begin{aligned} [\![\varphi]\!]^{\mathcal{M}} &= \{v \in W \mid \mathcal{M}, v \models \varphi\} \\ \mathbf{Min}_{\leqslant_w}(S) &= \{v \in S \cap W_w \mid \neg \exists u \in S \cap W_w \colon u <_w v\} \end{aligned}$$

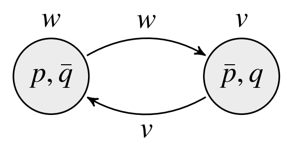
($\Rightarrow$) By contraposition. Suppose that there is a nonemtpy $S \subseteq W$ and a $w \in W$ such that $|\mathbf{Min}_{\leqslant_w}(S)| \neq 1$. Since $\leqslant_w$ is well-founded and $S \neq \varnothing$, $\mathbf{Min}_{\leqslant_w}(S) \neq \varnothing$, so $|\mathbf{Min}_{\leqslant_w}(S)| > 1$. Pick an arbitrary $v \in \mathbf{Min}_{\leqslant_w}(S)$, and let $V(p) = S$ and $V(q) = \{v\}$. Thus, $v \in \mathbf{Min}_{\leqslant_w}(S)$ but $v \notin [\![\neg q]\!]^{\mathcal{M}}$. But since $|\mathbf{Min}_{\leqslant_w}(S)| > 1$, there's a $u \in \mathbf{Min}_{\leqslant_w}(S)$ but $u \notin [\![q]\!]^{\mathcal{M}}$. So neither $\mathbf{Min}_{\leqslant_w}(S) \subseteq [\![q]\!]^{\mathcal{M}}$ nor $\mathbf{Min}_{\leqslant_w}(S) \subseteq [\![\neg q]\!]^{\mathcal{M}}$. Hence, $\mathcal{M}, w \not\models (p \mathrel{\Box\!\!\rightarrow} q) \vee (p \mathrel{\Box\!\!\rightarrow} \neg q)$.

($\Leftarrow$) Suppose for all nonempty $S \subseteq W$ and all $w \in W$, $|\mathbf{Min}_{\leqslant_w}(S)| = 1$. That means for all models $\mathcal{M}$ based on $\mathcal{F}$, $\left|\mathbf{Min}_{\leqslant_w}\left([\![p]\!]^{\mathcal{M}}\right)\right| = \{v\}$ for some $v \in W$. Now, either $v \in [\![q]\!]^{\mathcal{M}}$ or $v \in [\![\neg q]\!]^{\mathcal{M}}$. If the former, then $\mathbf{Min}_{\leqslant_w}(S) \subseteq [\![q]\!]^{\mathcal{M}}$, so $\mathcal{M}, w \models p \mathrel{\Box\!\!\rightarrow} q$. If the latter, $\mathbf{Min}_{\leqslant_w}(S) \subseteq [\![\neg q]\!]^{\mathcal{M}}$, so $\mathcal{M}, w \models p \mathrel{\Box\!\!\rightarrow} \neg q$. So either way, $\mathcal{M}, w \models (p \mathrel{\Box\!\!\rightarrow} q) \vee (p \mathrel{\Box\!\!\rightarrow} \neg q)$.

3. For each formula below, determine whether or not that formula is valid on Lewis's semantics. If the formula is valid, prove it by showing that every pointed model which makes the antecedent true makes the consequent true. If the formula is not valid, construct a pointed model that falsifies it. (Assume the ordering relations are well-founded and total, so that you can use the truth definition on slide 13. Also assume every $\leqslant_w$ is weakly centered.)

(a) $\alpha \mathrel{\square\!\!\rightarrow} (\beta \mathrel{\square\!\!\rightarrow} \alpha)$

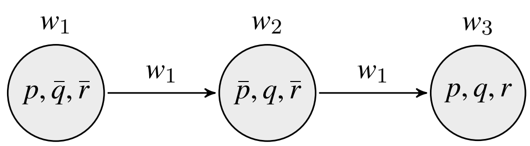
▶ Solution: Not valid. Consider the model below.



Now, $[\![q \mathrel{\square\!\!\rightarrow} p]\!]^{\mathcal{M}} = \varnothing$. But $\mathbf{Min}_{\leqslant_w}\left([\![p]\!]^{\mathcal{M}}\right) = \{w\}$, so $\mathbf{Min}_{\leqslant_w}\left([\![p]\!]^{\mathcal{M}}\right) \nsubseteq [\![q \mathrel{\square\!\!\rightarrow} p]\!]^{\mathcal{M}}$. Hence, $\mathcal{M}, w \not\models p \mathrel{\square\!\!\rightarrow} (q \mathrel{\square\!\!\rightarrow} p)$.

(b) $((\alpha \wedge \beta) \mathrel{\square\!\!\rightarrow} \gamma) \rightarrow (\alpha \mathrel{\square\!\!\rightarrow} (\beta \mathrel{\square\!\!\rightarrow} \gamma))$

▶ Solution: Not valid. Consider the model below.



The other orderings are not pictured, since they're not relevant. Now, $\mathcal{M}, w_1 \models (p \wedge q) \mathrel{\square\!\!\rightarrow} r$, since $\mathbf{Min}_{\leqslant_{w_1}}\left([\![p \wedge q]\!]^{\mathcal{M}}\right) = \{w_3\} \subseteq [\![r]\!]^{\mathcal{M}} = \{w_3\}$. But $w_1 \notin [\![q \mathrel{\square\!\!\rightarrow} r]\!]^{\mathcal{M}}$, since $\mathbf{Min}_{\leqslant_{w_1}}\left([\![q]\!]^{\mathcal{M}}\right) = \{w_2\} \nsubseteq [\![r]\!]^{\mathcal{M}} = \{w_3\}$. And yet $\mathbf{Min}_{\leqslant_{w_1}}\left([\![p]\!]^{\mathcal{M}}\right) = \{w_1\}$ since $\mathcal{M}, w_1 \models p$. So $\mathbf{Min}_{\leqslant_{w_1}}\left([\![p]\!]^{\mathcal{M}}\right) \nsubseteq [\![q \mathrel{\square\!\!\rightarrow} r]\!]^{\mathcal{M}}$, and thus $\mathcal{M}, w \not\models p \mathrel{\square\!\!\rightarrow} (q \mathrel{\square\!\!\rightarrow} r)$.

(c) $(((\alpha \wedge \beta) \mathrel{\square\!\!\rightarrow} \gamma) \wedge (\alpha \mathrel{\square\!\!\rightarrow} \beta)) \rightarrow (\alpha \mathrel{\square\!\!\rightarrow} \gamma)$

▶ Solution: Valid. Suppose $\mathcal{M}, w \models ((\alpha \wedge \beta) \mathrel{\square\!\!\rightarrow} \gamma) \wedge (\alpha \mathrel{\square\!\!\rightarrow} \beta)$. That means the following:

- $\mathbf{Min}_{\leqslant_w}\left([\![\alpha \wedge \beta]\!]^{\mathcal{M}}\right) \subseteq [\![\gamma]\!]^{\mathcal{M}}$
- $\mathbf{Min}_{\leqslant_w}\left([\![\alpha]\!]^{\mathcal{M}}\right) \subseteq [\![\beta]\!]^{\mathcal{M}}$.

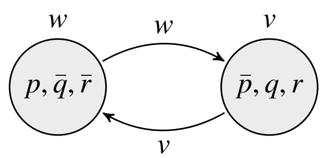
Now, let $v \in \mathbf{Min}_{\leqslant_w}\left([\![\alpha]\!]^{\mathcal{M}}\right)$ be arbitrary. It suffices to show that $v \in \mathbf{Min}_{\leqslant_w}\left([\![\alpha \wedge \beta]\!]^{\mathcal{M}}\right)$, for then by the first bullet above, $v \in [\![\gamma]\!]^{\mathcal{M}}$.

Clearly, since $v \in \mathbf{Min}_{\leqslant_w}\left([\![\alpha]\!]^{\mathcal{M}}\right)$, by the second bullet above, $v \in [\![\beta]\!]^{\mathcal{M}}$. So we just need to show that there is no $u \in [\![\alpha \wedge \beta]\!]^{\mathcal{M}}$ such that

$u <_w v$. Suppose for *reductio* that there was such a u. Then there is a $u \in [\![\alpha]\!]^{\mathcal{M}}$ such that $u <_w v$, which can't be since $v \in \mathbf{Min}_{\leqslant_w}\left([\![\alpha]\!]^{\mathcal{M}}\right)$, contradiction. So there is no $u \in [\![\alpha \wedge \beta]\!]^{\mathcal{M}}$ such that $u <_w v$. Hence, $v \in \mathbf{Min}_{\leqslant_w}\left([\![\alpha \wedge \beta]\!]^{\mathcal{M}}\right)$, as desired.

(d) $(\alpha \mathrel{\square\!\!\rightarrow} (\beta \mathrel{\square\!\!\rightarrow} \gamma)) \rightarrow (\beta \mathrel{\square\!\!\rightarrow} (\alpha \mathrel{\square\!\!\rightarrow} \gamma))$

► Solution: Not valid. Consider the model below.



Now, $[\![q \mathrel{\square\!\!\rightarrow} r]\!]^{\mathcal{M}} = W$, so $\mathbf{Min}_{\leqslant_w}\left([\![p]\!]^{\mathcal{M}}\right) = \{w\} \subseteq [\![q \mathrel{\square\!\!\rightarrow} r]\!]^{\mathcal{M}}$. Hence, $\mathcal{M}, w \models p \mathrel{\square\!\!\rightarrow} (q \mathrel{\square\!\!\rightarrow} r)$. But $[\![p \mathrel{\square\!\!\rightarrow} r]\!]^{\mathcal{M}} = \varnothing$, so $\mathbf{Min}_{\leqslant_w}\left([\![q]\!]^{\mathcal{M}}\right) = \{v\} \nsubseteq [\![p \mathrel{\square\!\!\rightarrow} r]\!]$. Hence, $\mathcal{M}, w \not\models q \mathrel{\square\!\!\rightarrow} (p \mathrel{\square\!\!\rightarrow} r)$.