Compactness and Löwenheim-Skolem

Phil 140A | Spring 2016

1. Suppose a sentence σ is true in every infinite model. Show that there's an $n \in \mathbb{N}$ such that σ is true in every model of at least size n.

2. Let $\varphi(x)$ be a formula. For any model $\mathfrak{A}$, define $\varphi(\mathfrak{A}) := \{a \in A \mid \mathfrak{A} \vDash \varphi(\bar{a})\}$ (the set of elements in $\mathfrak{A}$ satisfying φ). Suppose for each $n \in \mathbb{N}$, there is a model $\mathfrak{A}_n$ such that $|\varphi(\mathfrak{A}_n)| \geqslant n$. Show that there's a model $\mathfrak{A}$ such that $|\varphi(\mathfrak{A})| \geqslant \aleph_0$ (i.e., $\varphi(\mathfrak{A})$ is infinite).

3. Suppose there's a model $\mathfrak{A}$ such that $\varphi(\mathfrak{A})$ is infinite. Show that for any infinite cardinal number λ, there's a model $\mathfrak{B}$ such that $\mathfrak{B}$ satisfies the same sentences as $\mathfrak{A}$ and $\varphi(\mathfrak{B})$ is exactly of size λ.

4. Let P and Q be unary-predicates. Show that there is no first-order sentence μ that is true in a model $\mathfrak{A}$ iff $|P(\mathfrak{A}) \cap \neg Q(\mathfrak{A})| < |P(\mathfrak{A}) \cap Q(\mathfrak{A})|$ (i.e., most Ps are Qs in $\mathfrak{A}$).